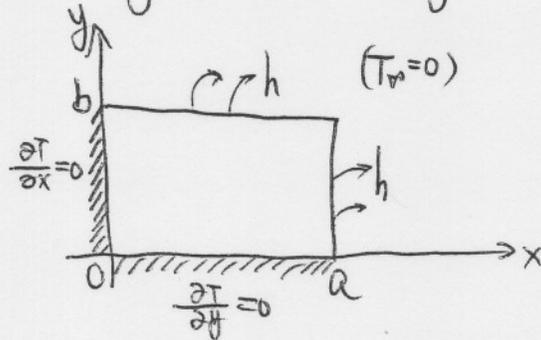


* Example 2 — Multi-dimensional transient conduction.

Consider a long bar of rectangular cross-section as shown below. The initial temperature distribution is $f(x, y)$. Find temperature distribution at any time and any location in the bar.



The complete problem: (2D)

$$\frac{\partial^2 T(x, y, t)}{\partial x^2} + \frac{\partial^2 T(x, y, t)}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T(x, y, t)}{\partial t}$$

B.C. $\left\{ \begin{array}{l} \frac{\partial T}{\partial x} \Big|_{x=0} = 0 \\ \frac{\partial T}{\partial y} \Big|_{y=0} = 0 \\ -k \frac{\partial T}{\partial x} \Big|_{x=a} = h T \Big|_{x=a} \\ -k \frac{\partial T}{\partial y} \Big|_{y=b} = h T \Big|_{y=b} \end{array} \right.$ Homogeneous B.C.

I.C. $T \Big|_{t=0} = f(x, y)$

① Separation of $T(x, y, t)$:

Assume, $T(x, y, t) = X(x) Y(y) \Gamma(t)$

(three-factor product expression)

Then: $\frac{X''}{X} + \frac{Y''}{Y} = \underbrace{\frac{1}{\alpha} \frac{\Gamma'}{\Gamma}}_{\Gamma(x) = C e^{-\lambda^2 \alpha x}} = -\lambda^2 \quad (\alpha > 0)$

\parallel \parallel
 μ_1 μ_2

$$\left\{ \begin{array}{l} \frac{X''}{X} = \mu_1 \\ \frac{Y''}{Y} = \mu_2 \end{array} \right\} \Rightarrow \mu_1 + \mu_2 = -\lambda^2$$

② Solving the ODEs

<1> Look at the problem for $X(x)$:

$$X'' - \mu_1 X = 0$$

B.C. $\left\{ \begin{array}{l} \frac{dX}{dx} \Big|_{x=0} = 0 \\ -k \frac{dX}{dx} \Big|_{x=a} = hX \Big|_{x=a} \end{array} \right.$ (homogeneous B.C.)

The eigenvalue problem requires: $\mu_1 < 0$

Let: $\mu_1 = -\beta^2$

Therefore: $X'' + \beta^2 X = 0$

$X(x) = A \cos \beta x$ using B.C. $\frac{dX}{dx} \Big|_{x=0} = 0$.

Imposing B.C. $-k \frac{dX}{dx} \Big|_{x=a} = hX \Big|_{x=a}$

$$-k(-A\beta \sin \beta a) = hA \cos \beta a$$

$$A(k\beta \sin \beta a - h \cos \beta a) = 0$$

$$\boxed{\cot \beta a = \frac{k\beta}{h}} \quad \beta = \beta_m, m=1, 2, 3, \dots$$

eigenvalues.

so for each m : $\Sigma_m(x) = A_m \cos \beta_m x$

<2> Look at the problem for $Y(y)$:

$$Y'' - \mu_2 Y = 0$$

B.C. $\begin{cases} \frac{dY}{dy}|_{y=0} = 0 \\ -k \frac{dY}{dy}|_{y=b} = hY|_{y=b} \end{cases}$ (homogeneous B.C.)

The eigenvalue problem requires: $\mu_2 < 0$

Let: $\mu_2 = -\nu^2$

Therefore: $Y'' + \nu^2 Y = 0$

$Y(y) = B \cos \nu y$ using B.C. $\frac{dY}{dy}|_{y=0} = 0$

Imposing B.C. $-k \frac{dY}{dy}|_{y=b} = hY|_{y=b}$

$$-k(-B\nu \sin \nu b) = hB \cos \nu b$$

$$B(k\nu \sin \nu b - h \cos \nu b) = 0$$

$$\boxed{\cot \nu b = \frac{k\nu}{h}}$$

$\nu = \nu_n \quad n=1, 2, 3, \dots$
eigenvalues.

so for each n :

$Y_n(y) = B_n \cos \nu_n y$

③ Making final solution.

Note: $\mu_1 + \mu_2 = -\lambda^2 \Rightarrow -\beta^2 - \gamma^2 = -\lambda^2$

so: $\Gamma(t) = C e^{-(\beta^2 + \gamma^2)xt}$

and for every "m" and "n":

$T_{mn}(x, y, t) = C_{mn} \cos \beta_m x \cos \gamma_n y e^{-(\beta_m^2 + \gamma_n^2)xt}$

so: $T(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos \beta_m x \cos \gamma_n y e^{-(\beta_m^2 + \gamma_n^2)xt}$

with β_m defined by $\cot \beta_m a = \frac{k\beta_m}{h}$

γ_n defined by $\cot \gamma_n b = \frac{k\gamma_n}{h}$

④ Determining unknown coefficient.

Applying initial condition: $T|_{t=0} = f(x, y)$

so: $f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos \beta_m x \cos \gamma_n y$

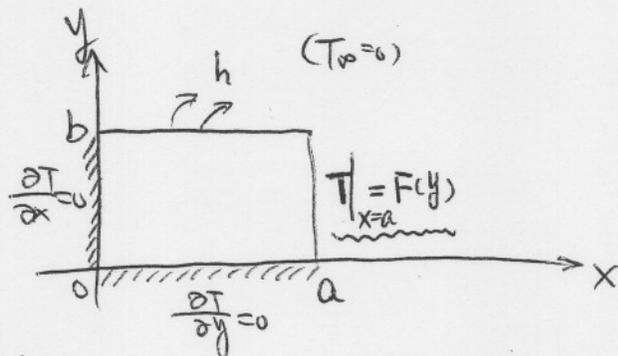
Multiplying $\cos \beta_{m'} x \cos \gamma_{n'} y$ and integrating over $\int_0^a \int_0^b$, and using the orthogonal property:

$C_{mn} = \frac{\int_{y=0}^b \int_{x=0}^a f(x, y) \cos \beta_m x \cos \gamma_n y dx dy}{\int_{x=0}^a \cos^2 \beta_m x dx \cdot \int_{y=0}^b \cos^2 \gamma_n y dy}$

Note: If $f(x, y) = f_1(x) f_2(y)$, then: $C_{mn} = D_m \cdot E_n$.
(separable) (separable)

*Example 3 — Transient problem with nonhomogeneous B.C.

Consider the same transient problem as in example 2, except the temperature distribution $T(x, y, t)$ is maintained as $F(y)$ at $x=a$:



The complete problem (2D)

$$\frac{\partial^2 T(x, y, t)}{\partial x^2} + \frac{\partial^2 T(x, y, t)}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T(x, y, t)}{\partial t}$$

B.C.

$$\begin{cases} \frac{\partial T}{\partial x} \Big|_{x=0} = 0 \\ T \Big|_{x=a} = F(y) \end{cases} \leftarrow \text{Nonhomogeneous (not a function of time)}$$

$$\begin{cases} \frac{\partial T}{\partial y} \Big|_{y=0} = 0 \\ -k \frac{\partial T}{\partial y} \Big|_{y=b} = h T \Big|_{y=b} \end{cases}$$

I.C.

$$T \Big|_{t=0} = f(x, y)$$

While the temperature will vary with time, it will none the less approach a steady state eventually. The solution consists of a transient part and a steady-state part:

$$\underline{T(x, y, t) = T_h(x, y, t) + T_I(x, y)}$$

\uparrow \uparrow
 transient steady-state

with $T_h(x, y, t)$ satisfying:

$$\frac{\partial^2 T_h(x, y, t)}{\partial x^2} + \frac{\partial^2 T_h(x, y, t)}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T_h(x, y, t)}{\partial t}$$

B.C. $\begin{cases} \frac{\partial T_h}{\partial x} \Big|_{x=0} = 0 \\ T_h \Big|_{x=a} = 0 \end{cases}$

$\begin{cases} \frac{\partial T_h}{\partial y} \Big|_{y=0} = 0 \\ -k \frac{\partial T_h}{\partial y} \Big|_{y=b} = h T_h \Big|_{y=b} \end{cases}$

I.C. $T_h \Big|_{t=0} = f(x, y) - T_I(x, y)$

← homogeneous

and $T_I(x, y)$ satisfying:

$$\frac{\partial^2 T_I(x, y)}{\partial x^2} + \frac{\partial^2 T_I(x, y)}{\partial y^2} = 0$$

B.C. $\begin{cases} \frac{\partial T_I}{\partial x} \Big|_{x=0} = 0 \\ T_I \Big|_{x=a} = F(y) \end{cases}$

$\begin{cases} \frac{\partial T_I}{\partial y} \Big|_{y=0} = 0 \\ -k \frac{\partial T_I}{\partial y} \Big|_{y=b} = h T_I \Big|_{y=b} \end{cases}$

steady-state!

← nonhomogeneous

For $T_h(x, y, t)$:

$$T_h(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos \beta_m x \cos \gamma_n y e^{-(\beta_m^2 + \gamma_n^2) \alpha t}$$

with: $\begin{cases} \beta_m \text{ is defined by: } \cos \beta_m a = 0 \\ \gamma_n \text{ is defined by: } \cot \gamma_n b = \frac{k}{h} \gamma_n \end{cases}$

For $T_I(x, y)$:

$$T_I(x, y) = \sum_{l=1}^{\infty} D_l \cosh \lambda_l x \cos \lambda_l y$$

with: $\lambda_l \text{ is defined by: } \cot \lambda_l b = \frac{k}{h} \lambda_l$

and $D_l = \frac{\int_0^b F(y) \cos \lambda_l y dy}{\cosh \lambda_l a \cdot \int_0^b \cos^2 \lambda_l y dy}$

3.3. Separation of Variables — Cylindrical system

* Example

Consider a solid cylinder of length $2L$ and diameter $2R_0$. The initial temperature is uniform and constant at T_i .

At $t=0$, the cylinder is immersed in a fluid of temperature T_∞ . The convective heat transfer coefficient is h for all surfaces.

Determine the temperature distribution.

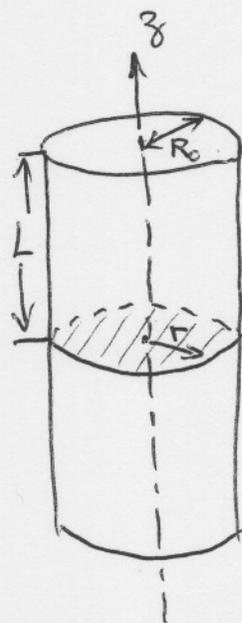
The complete problem: (2D)

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

B.C. $\left\{ \begin{array}{l} T|_{r=0} = \text{finite} \\ -k \frac{\partial T}{\partial r} \Big|_{r=R_0} = h(T|_{r=R_0} - T_\infty) \end{array} \right.$

$\left\{ \begin{array}{l} \frac{\partial T}{\partial z} \Big|_{z=0} = 0 \quad (\text{from symmetry}) \\ -k \frac{\partial T}{\partial z} \Big|_{z=L} = h(T|_{z=L} - T_\infty) \end{array} \right.$

I.C. $T|_{t=0} = T_i$



The conduction problem defined by the above equation plus boundary and initial conditions is for the upper half of the solid cylinder. We have used the symmetry to write the boundary condition at $z=0$: $T(r, -z, t) = T(r, z, t)$.

Note: T_{∞} is a constant, so let $\theta(r, z, t) \equiv T(r, z, t) - T_{\infty}$

so:
$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right) + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t}$$

B.C.
$$\left\{ \begin{array}{l} \theta|_{r=0} = \text{finite} \\ -k \frac{\partial \theta}{\partial r} \Big|_{r=R_0} = h \theta|_{r=R_0} \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial \theta}{\partial z} \Big|_{z=0} = 0 \\ -k \frac{\partial \theta}{\partial z} \Big|_{z=L} = h \theta|_{z=L} \end{array} \right.$$

I.C.
$$\theta|_{t=0} = T_i - T_{\infty} \equiv \theta_i$$

all homogeneous BC.

Solution:

① Separation of $\theta(r, z, t)$:

Assume: $\theta(r, z, t) = R(r) Z(z) \Gamma(t)$

therefore:
$$\frac{\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right)}{R} + \frac{Z''}{Z} = \frac{\frac{1}{\alpha} \frac{d\Gamma}{dt}}{\Gamma} = -\lambda^2$$

$$\Gamma(t) = A e^{-\lambda^2 \alpha t}$$

$$\left\{ \begin{array}{l} \frac{\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right)}{R} = \nu \\ \frac{Z''}{Z} = \mu \end{array} \right.$$

with:
$$\mu + \nu = -\lambda^2$$

② Solving ODEs.

(1) For $Z(z)$:

$$\boxed{\begin{aligned} Z'' - \mu Z &= 0 \\ \text{B.C. } \left. \begin{aligned} \frac{dZ}{dz} \Big|_{z=0} &= 0 \\ -k \frac{dZ}{dz} \Big|_{z=L} &= hZ \Big|_{z=L} \end{aligned} \right\} \end{aligned}}$$

The eigenvalue problem (homogeneous B.C.s) requires:

$$\underline{\mu = -\beta^2} \quad (\mu < 0)$$

$$\text{So: } \underline{Z'' + \beta^2 Z = 0}$$

$$\text{general solution: } \underline{Z(z) = C \cos \beta z + D \sin \beta z}$$

$$\begin{aligned} \text{Imposing B.C. } \frac{dZ}{dz} \Big|_{z=0} = 0 &\Rightarrow D = 0 \\ &\Rightarrow \underline{Z(z) = C \cos \beta z.} \end{aligned}$$

$$\begin{aligned} \text{Imposing B.C. } -k \frac{dZ}{dz} \Big|_{z=L} = hZ \Big|_{z=L} &\Rightarrow -k(-C\beta \sin \beta L) = hC \cos \beta L \\ \text{i.e.: } \boxed{\cot \beta L = \frac{k\beta}{h}} & \end{aligned}$$

Which determines the eigenvalues of β : $\underline{\beta = \beta_m, m=1, 2, 3, \dots}$

(2) For $R(r)$:

$$\boxed{\begin{aligned} \frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \nu R &= 0 \\ \text{B.C. } \left. \begin{aligned} R \Big|_{r=0} &= \text{finite} \\ -k \frac{dR}{dr} \Big|_{r=R_0} &= hR \Big|_{r=R_0} \end{aligned} \right\} \end{aligned}}$$

The eigenvalue problem (homogeneous B.C.s) requires:

$$\underline{\nu = -\gamma^2 \quad (\nu < 0)}$$

$$\text{SO: } \underline{R'' + \frac{1}{r}R' + \gamma^2 R = 0}$$

$$\text{general solution: } \underline{R(r) = E J_0(\gamma r) + F Y_0(\gamma r)}$$

$$\underline{\text{Imposing B.C. } R|_{r=0} = \text{finite} \Rightarrow F = 0 \quad (Y_0(\xi) \text{ diverges for } \xi = 0)}$$

$$\Rightarrow \underline{R(r) = E J_0(\gamma r)}$$

$$\underline{\text{Imposing B.C. } -k \frac{dR}{dr} \Big|_{r=R_0} = h R|_{r=R_0} \Rightarrow -k \cdot E \gamma J_0'(\gamma R_0) = h \cdot E J_0(\gamma R_0)}$$

$$\text{i.e.: } \boxed{\frac{J_0(\gamma R_0)}{J_0'(\gamma R_0)} = -\frac{k \gamma}{h}}$$

Which determines the eigenvalues of γ : $\underline{\gamma = \gamma_n, \quad n = 1, 2, 3, \dots}$

③ Making final solution.

$$\text{For each } m: \underline{Z_m(z) = C_m \cos \beta_m z}$$

$$\text{with } \beta_m \text{ defined by: } \cot \beta_m L = \frac{k \beta_m}{h}$$

$$\text{For each } n: \underline{R_n(r) = E_n J_0(\gamma_n r)}$$

$$\text{with } \gamma_n \text{ defined by: } \frac{J_0(\gamma_n R_0)}{J_0'(\gamma_n R_0)} = -\frac{k \gamma_n}{h}$$

and: $\lambda_{mn}^2 = \beta_m^2 + \gamma_n^2$

Therefore, for every m and n :

$$\theta_{mn}(r, z, t) = A_{mn} J_0(\gamma_n r) \cos \beta_m z e^{-(\beta_m^2 + \gamma_n^2) \alpha t}$$

and:

$$\theta(r, z, t) = \sum_{m,n}^{\infty} A_{mn} J_0(\gamma_n r) \cos \beta_m z e^{-(\beta_m^2 + \gamma_n^2) \alpha t}$$

④ Determining unknown coefficient.

Applying initial condition: $\theta|_{t=0} = \theta_i = T_i - T_{\infty}$

$$\text{so: } \theta_i = T_i - T_{\infty} = \sum_{m,n} A_{mn} J_0(\gamma_n r) \cos \beta_m z$$

therefore:

$$A_{mn} = \frac{(T_i - T_{\infty}) \int_0^{R_0} J_0(\gamma_n r) r dr \int_0^L \cos \beta_m z dz}{\int_0^{R_0} J_0^2(\gamma_n r) r dr \int_0^L \cos^2 \beta_m z dz}$$

$$T(r, z, t) = T_{\infty} + \sum_{m,n}^{\infty} A_{mn} J_0(\gamma_n r) \cos \beta_m z e^{-(\beta_m^2 + \gamma_n^2) \alpha t}$$